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Energy-conserving Galerkin representation of clamped plates under a moderately large deflection

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Abstract

In the moderately large deflection plate theory of von Karman and Chu–Herrmann, one can formulate dynamic equations of a thin plate by considering either the transverse and in-plane displacements, *w–u–v* formulation, or the transverse displacement and Airy function, *w–F* formulation. Previously, for a simply supported plate we have investigated the Hamiltonian property of modal equations obtained by the Galerkin representation under *w–u–v* and *w–F* formulations. We extend here such investigations to a rectangular clamped plate with similar conclusions. That is, the modal equations of *w–F* formulation are Hamiltonian and hence energy conserving at any order of truncation. On the other hand, the corresponding modal equations of *w–u–v* formulation do not conserve energy when only a small number of sine terms are included in the in-plane displacement expansions.

1. Introduction

The classical linear plate theory deals exclusively with the inertia and plate bending of transverse displacement w. Beyond that, the so-called non-linear theory of von Karman and Chu–Herrmann [1] incorporates the first order effects of in-plane displacements u and v, which give rise to membrane stretching. The three displacement equations for w, u, and v are called w–u–v formulation of a moderately large-deflection plate theory [2]. For a thin plate, it is customary to ignore the inertia of in-plane displacements in comparison to the transverse motion. This then permits us to replace the two in-plane static equations for u and v with a compatibility relation for the Airy stress function F, and hence the alternate w–F formulation. Although the w–u–v and w–F formulations are equivalent in theory, they are not in practice. This is because w, u, v, and F must all be expressed by a finite degree-of-freedom representation by whatever the

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means chosen for actual computations, such as the Galerkin representation, finite difference, and finite element. In Ref. [3], we have compared the modal equations of two plate formulations derived by the Galerkin procedure for a simply supported plate, and the intent of this paper is to extend such a comparison to a rectangular clamped plate.

Basic equations of the two plate formulations are briefly summarized (Section 2) in a dimensionless form for uniformity. Since the plate equations are couched in Hamilton's variational principle, the total kinetic and strain (potential) energy must be conserved independent of the representation for plate dynamics. In the *w–F* formulation, the modal equations of Galerkin representation are Hamiltonian, which is a stronger dynamical property than merely energy conserving. For the simply supported plate [3], plate modes can be formed by a product of simply supported beam modes which are sine functions. On the other hand, such a simple construction does not work for a clamped plate, hence the products of clamped-beam modes are called pseudo-plate modes (Section 3). We can however express the plate modes of clamped plate by a linear combination of the pseudo-plate modes, and this is what complicates the analysis of a clamped plate. Nonetheless, the upshot of the present analysis parallels the simply supported plate case [3]. That is, in the *w–F* formulation, modal equations are Hamiltonian and hence energy conserving (Section 4), whereas the modal equations of *w–u–v* formulation do not conserve energy when only a dozen of sine terms is included in the in-plane displacement expansions (Section 5). Previously we have presented a gradual approach to energy conservation of the *w–u–v* formulation by successively including more and more sine terms in the in-plane displacement expansions [3]. However, such successive analyses are not feasible to carry out here, for they involve too excessive algebraic manipulations.

2. Synopsis of plate equations in dimensionless form

Summarized in Ref. [3] are basic equations of the von Karman–Chu–Herrmann plate theory for a moderately large deflection. In three plate displacements (u, v, w) along the usual Cartesian co-ordinates (x, y, z), the classical linear theory considers the inertia and plate bending due to transverse displacement w only. The non-linear plate formulation of von Karman and Chu–Herrmann [1] includes the first- order geometric non-linearity of membrane stretching by the in-plane displacements u and v. For a thin plate, one may ignore the inertia of in-plane displacement motion along with the rotatory effects [2]. Then, the static in-plane displacement equations for (u, v) and transverse displacement equation for w constitute the *w–u–v* formulation of a moderately large deflection plate. Instead, by combining the static constraints on u and v into a compatibility relation for Airy function F, one ends up with only the transverse displacement equation for w and compatibility relation for F, the so-called *w–F* formulation. For uniformity, we have normalized in Ref. [3] the plate co-ordinates (x, y) by the plate side L_x in x, transverse displacement w by the plate thickness h, in-plane displacements (u, v) by h^2/L_x, Airy stress function F by D, the stress resultants (N_x, N_y, N_{xy}) by D/L_x^2, and time t by $L_x^2/\pi^2\sqrt{\rho h/D}$. Here, $D = Eh^3/12(1-v^2)$ is the flexural rigidity, E denotes Young's modulus of elasticity, ρ is the mass density, and v is the Poisson' ratio. With these normalizations, we find that only the aspect ratio $r = L_x/L_y$ (where L_y is the plate side in y) and v would show up in the dimensionless plate equations to be presented.

2.1. The w–u–v formulation

The dynamic equation for transverse displacement becomes in dimensionless form

$$\frac{\partial^2 w}{\partial t^2} + \frac{1}{\pi^4}\left(\frac{\partial^4 w}{\partial x^4} + 2r^2 \frac{\partial^4 w}{\partial x^2\,\partial y^2} + r^4 \frac{\partial^2 w}{\partial y^4}\right) = \frac{12}{\pi^4}\left(N_x \frac{\partial^2 w}{\partial x^2} + 2rN_{xy} \frac{\partial^2 w}{\partial x\,\partial y} + r^2 N_y \frac{\partial^2 w}{\partial y^2}\right), \tag{1}$$

in which the stress resultants (N_x, N_y, N_{xy}) are governed by the following static equations:

$$\frac{\partial N_x}{\partial x} + r\frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_{xy}}{\partial x} + r\frac{\partial N_y}{\partial y} = 0. \tag{2}$$

Here, $N_x = \partial u/\partial x + vr\partial v/\partial y + \frac{1}{2}(\partial w/\partial x)^2 + \frac{1}{2}vr^2(\partial w/\partial y)^2$, $N_y = v\,\partial u/\partial x + r\partial v/\partial y + \frac{1}{2}v(\partial w/\partial x)^2 + \frac{1}{2}r^2(\partial w/\partial y)^2$ and $N_{xy} = \frac{1}{2}(1-v)(r\partial u/\partial y + \partial v/\partial x + r(\partial w/\partial x)(\partial w/\partial y))$. By writing out Eq. (2) in detail, we have the usual in-plane displacement equations but without the inertia terms:

$$\frac{\partial^2 u}{\partial x^2} + d_1 r^2 \frac{\partial^2 u}{\partial y^2} + d_2 r \frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + d_2 r^2 \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + d_1 r^2 \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = 0,$$

$$d_1 \frac{\partial^2 v}{\partial x^2} + r^2 \frac{\partial^2 v}{\partial y^2} + d_2 r \frac{\partial^2 u}{\partial x\,\partial y} + d_1 r \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + d_2 r \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\,\partial y} + r^3 \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} = 0, \tag{3}$$

where $d_1 = (1-v)/2$ and $d_2 = (1+v)/2$. Note that Eqs. (1)–(3) are identical to Eqs. (10)–(12) of Ref. [3].

Although Eqs. (1) and (2) are the starting point of discussion in this paper, it is important to recall that they have been derived by invoking Hamilton's variational principle. This involves the expression for kinetic energy,

$$U_k = \frac{1}{2}\int_0^1 \int_0^1 \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y, \tag{4}$$

and strain energy which we split into the bending strain energy,

$$U_b = \frac{1}{2\pi^4}\int_0^1 \int_0^1 \left(\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2vr^2 \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + r^4\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-v)r^2\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right)\mathrm{d}x\,\mathrm{d}y, \tag{5}$$

and membrane stretching strain energy,

$$U_m = \frac{6}{\pi^4}\int_0^1 \int_0^1 \left(\varepsilon_x^2 + 2v\varepsilon_x\varepsilon_y + \varepsilon_y^2 + \frac{(1-v)}{2}\varepsilon_{xy}^2\right)\mathrm{d}x\,\mathrm{d}y, \tag{6}$$

where $\varepsilon_x = \partial u/\partial x + \frac{1}{2}(\partial w/\partial x)^2$, $\varepsilon_y = r\partial v/\partial y + \frac{1}{2}r^2(\partial w/\partial y)^2$, and $\varepsilon_{xy} = r\partial u/\partial y + \partial v/\partial x + r(\partial w/\partial x)(\partial w/\partial y)$. Note that Eqs. (4)–(6) are also the dimensionless energy expressions.

2.2. The w–F formulation

By virtue of $N_x = r^2\partial^2 F/\partial y^2$, $N_y = \partial^2 F/\partial x^2$ and $N_{xy} = -r\partial^2 F/\partial x\,\partial y$, we can rewrite the transverse displacement equation (1) in terms of the Airy stress function F,

$$\frac{\partial^2 w}{\partial t^2} + \frac{1}{\pi^4}\left(\frac{\partial^4 w}{\partial x^4} + 2r^2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + r^4\frac{\partial^2 w}{\partial y^4}\right) = \frac{12r^2}{\pi^4}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}\right). \tag{7}$$

Moreover, two members of Eq. (2) are combined to give a single compatibility relation of the fourth order,

$$r^{-2}\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2\,\partial y} + r^2\frac{\partial^2 F}{\partial y^4} = (1-v^2)\left(\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right). \tag{8}$$

Note that Eqs. (7) and (8) are Eqs. (17) and (18) of Ref. [3], to which the readers are referred for derivation. Although the expressions for kinetic energy (4) and bending strain energy (5) do not involve F, it is necessary to express the membrane stretching strain energy (6) in terms of F:

$$U_m = \frac{6}{\pi^4(1-v^2)}\int_0^1\int_0^1\left(\left(\frac{\partial^2 F}{\partial x^2}\right)^2 - 2vr^2\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + r^4\left(\frac{\partial^2 F}{\partial y^2}\right)^2 + 2(1+v)r^2\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2\right)\mathrm{d}x\,\mathrm{d}y. \tag{9}$$

A brief summary of thin-plate equations of the two plate formulations is presented here, and the readers are referred to Refs. [2,4–6] for detailed discussions.

3. Plate modal functions for a clamped plate

For the Galerkin representation, it is essential that one has access to plate modal functions, or simply called plate modes, as the basis functions to expand the transverse displacement. For the simply supported plate [3], we have expanded $w = \sum_{n=1,3}\sum_{m=1,3} a_{n,m}(t)\psi_n(x)\psi_m(y)$ in the first four symmetric plate modes, where $\psi_i(\xi) = \sqrt{2}\sin(i\pi\xi)$ are the normalized beam modes with the simply supported beam end conditions, $\psi_n = \psi_n'' = 0$ at $\xi = 0$ and 1. To simplify the notations, we let column vectors $\mathbf{q} = (q_1, q_2, q_3, q_4) \triangleq (a_{1,1}, a_{1,3}, a_{3,1}, a_{3,3})$ and $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \triangleq \{\psi_1(x)\psi_1(y), \psi_1(x)\psi_3(y), \psi_3(x)\psi_1(y), \psi_3(x)\psi_3(y)\}$. We then recast the summation into a scalar product $(\mathbf{q}\cdot\mathbf{\Phi})$, written as $\mathbf{q}^{\mathrm{T}}\mathbf{\Phi}$ also, where T is the transpose, and write the displacement expansion in a compact form,

$$w = (\mathbf{q}\cdot\mathbf{\Phi}) \equiv \mathbf{q}^{\mathrm{T}}\mathbf{\Phi}, \tag{10}$$

indicating that $q_i(t)$ are modal amplitudes of the plate modes $\Phi_i(x,y)$. Technically speaking, by plate modes we mean that $\Phi_i(x,y)$ are the eigenfunctions of biharmonic operator ∇^4, so that they are in fact plate-bending modes. It is indeed fortuitous that $\{\psi_n(x)\psi_m(y)\}$ are the actual plate modes of a rectangular simply supported plate. To see this, the inertial and plate bending terms of

Eqs. (1) and (7) give rise to a linear modal equation

$$\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \tag{11}$$

where the overhead dot denotes $\partial/\partial t$. That stiffness matrix $\mathbf{C}$ turns out diagonal implies $\boldsymbol{\Phi} = \{\psi_n(x)\psi_m(y)\}$ are the plate modes of a simply supported plate.

In an analogous fashion, let us construct plate modes $\boldsymbol{\Phi} = \{\phi_n(x)\phi_m(y)\}$ for a clamped plate by the normalized clamped-beam modes

$$\phi_i(\xi) = \cosh \ell_i\xi - \cos \ell_i\xi + \frac{(\cos \ell_i - \cosh \ell_i)}{(\sin \ell_i - \sinh \ell_i)}(\sin \ell_i\xi - \sinh \ell_i\xi), \tag{12}$$

where $\ell_1(\approx 4.730)$, $\ell_2(\approx 7.853)$, $\ell_3(\approx 10.996)$, $\ell_4(\approx 14.137)\ldots$ are the roots of $\cos \ell \cosh \ell = 1$ [7]. With Eq. (12), however, the stiffness matrix $\mathbf{C} = \{C_{i,j}^{true}\}$ is non-diagonal,

$$C_{i,j}^{true} = \pi^{-4} \times \begin{pmatrix} (1+r^4)\ell_1^4 + 2r^2b_1^2 & 2r^2b_1b_2 & 2r^2b_1b_2 & 2r^2b_2^2 \\ & \ell_1^4 + r^4\ell_3^4 + 2r^2b_1b_3 & 2r^2b_2^2 & 2r^2b_2b_3 \\ & & r^4\ell_1^4 + \ell_3^4 + 2r^2b_1b_3 & 2r^2b_2b_3 \\ & \textit{Symmetric} & & (1+r^4)\ell_3^4 + 2r^2b_3^2 \end{pmatrix}, \tag{13}$$

with $b_i = \ell_i(2(\cos \ell_i - \cosh \ell_i) + \ell_i(\sin \ell_i + \sinh \ell_i))/(\sin \ell_i - \sinh \ell_i)$ for $i = 1$ or 3 and $b_2 = 4\ell_1^2\ell_3^2(\chi(\ell_1, \ell_3) - \chi(\ell_3, \ell_1))/\{(\ell_1^4 - \ell_3^4)(\sin \ell_1 - \sinh \ell_1)(\sin \ell_3 - \sinh \ell_3)\}$, where $\chi(p, q) = p\{(\cos p - \cosh p)(\sin q - \sinh q) + \sin p \sinh p(\cosh q \sin q - \cos q \sinh q)\}$. Here, the superscript true in $C_{i,j}^{true}$ refers to that true clamped-beam modes $\phi_n(\xi)$ are used in Eq. (13).

Note that $\{\phi_n(x)\phi_m(y)\}$ do not represent the actual plate modes, and hence are called pseudo-plate modes of the clamped plate. There is however a standard procedure for constructing actual plate modes by a linear combination of $\{\phi_n(x)\phi_m(y)\}$. We begin by decomposing $\mathbf{C}$ into the eigenvectors $\mathbf{e}_i$ for eignvalues λ_i. Let us now introduce a column vector $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4)$ of new modal amplitudes Q_i, which are related to the old q_i by

$$\mathbf{q} = \mathbf{T}\mathbf{Q}, \tag{14}$$

where $\mathbf{T} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a matrix formed by the eigenvectors $\mathbf{e}_i$ in the columns. On the one hand, by introducing Eq. (14) into Eq. (11) we obtain a new modal equation $\ddot{\mathbf{Q}} + \boldsymbol{\lambda}\mathbf{Q} = 0$, where $\boldsymbol{\lambda}$ is a diagonal matrix with eigenvalues λ_i along its diagonal. On the other hand, by inserting Eq. (14) into Eq. (10) we have an alternate expansion

$$w = \mathbf{Q}^{\mathrm{T}}\boldsymbol{\Theta}, \tag{15}$$

where the column vector $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ has the components $\boldsymbol{\Theta}_i = (\mathbf{e}_i \cdot \boldsymbol{\Phi}(x, y))$. Thus, we have expressed actual plate modes $\boldsymbol{\Theta}$ in a linear combination of pseudo-plate modes $\boldsymbol{\Phi} = \{\phi_n(x)\phi_m(y)\}$, and the modal equation for $\mathbf{Q}$ now has a diagonal stiffness matrix. For a numerical illustration for $r = 0.9$,

$$C^{true}_{i,j} = \begin{pmatrix} 11.03 & -1.99 & -1.99 & 1.57 \\ & 123.83 & 1.57 & -16.0 \\ & & 173.67 & -16.0 \\ & Sym. & & 411.21 \end{pmatrix}$$

has eigenvectors $\mathbf{e}_1 = (0.999, 0.0171, 0.0118, -0.00278)$, $\mathbf{e}_2 = (-0.0167, 0.998, -0.0144, 0.0547)$, $\mathbf{e}_3 = (-0.0118, 0.0105, 0.997, 0.0677)$, and $\mathbf{e}_4 = (0.00451, -0.0555, -0.067, 0.996)$ for the eigenvalues $\lambda = (10.97, 122.97, 172.63, 413.18)$. Then, the actual plate modes are given by
$\Theta_1 = 0.999\phi_1(x)\phi_1(y) + 0.0171\phi_1(x)\phi_3(y) + 0.0118\phi_3(x)\phi_1(y) - 0.00278\phi_3(x)\phi_3(y)$,
$\Theta_2 = -0.0167\phi_1(x)\phi_1(y) + 0.998\phi_1(x)\phi_3(y) - 0.0144\phi_3(x)\phi_1(y) + 0.0547\phi_3(x)\phi_3(y)$,
$\Theta_3 = -0.0118\phi_1(x)\phi_1(y) + 0.0105\phi_1(x)\phi_3(y) + 0.997\phi_3(x)\phi_1(y) + 0.0677\phi_3(x)\phi_3(y)$,
$\Theta_4 = 0.00451\phi_1(x)\phi_1(y) - 0.0555\phi_1(x)\phi_3(y) - 0.067\phi_3(x)\phi_1(y) + 0.996\phi_3(x)\phi_3(y)$.

Since $(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \approx (0.999\phi_1(x)\phi_1(y), 0.998\phi_1(x)\phi_3(y), 0.997\phi_3(x)\phi_1(y), 0.996\phi_3(x)\phi_3(y))$, it is shown that $\{\phi_n(x)\phi_m(y)\}$ represent a good approximation to the actual clamped-plate modes Θ_i, although the approximation gets slightly worse with the increasing order of plate modes.

Instead of Eq. (12), the trigonometric functions $\varsigma_n(\xi) = \sin(n\pi\xi)\sin(\pi\xi) \equiv \frac{1}{2}(\cos(n-1)\pi\xi - \cos(n+1)\pi\xi)$ satisfying the end condition $\varsigma_n = \varsigma_n' = 0$ at $\xi = 0$ and 1 have been used exclusively in the literatures [5,8–11] to construct plate modes of a clamped plate for analytic expediency. Here, we form an orthonormal set of $\varsigma_n(\xi)$ by the Gram–Schmidt procedure [12]

$$\varphi_1 = \sqrt{\tfrac{8}{3}}\varsigma_1(\xi), \quad \varphi_2 = 2\varsigma_2(\xi), \quad \varphi_3 = \sqrt{\tfrac{24}{5}}\varsigma_3(\xi) + \sqrt{\tfrac{8}{15}}\varsigma_1(\xi),$$
$$\varphi_4 = \sqrt{\tfrac{16}{3}}\varsigma_4(\xi) + \sqrt{\tfrac{4}{3}}\varsigma_2(\xi), \ldots \qquad (16)$$

As shown in Fig. 1, $\varphi_n(\xi)$ are in a close agreement with $\phi_n(\xi)$ over the entire $\xi = (0, 1)$, except for where the peaks and troughs are. Hence, we shall simply call $\varphi_n(\xi)$ trigonometric clamped-beam modes in contrast to the true clamped-beam modes $\phi_n(\xi)$. Not surprisingly, the stiffness matrix $\mathbf{C} = \{C^{trig}_{i,j}\}$ based on trigonometric clamped-beam modes $\varphi_n(\xi)$ is again non-diagonal:

$$C^{trig}_{i,j} = \frac{16}{45} \times \begin{pmatrix} 5(3+2r^2+3r^4) & -2\sqrt{5}r^2(2+3r^2) & -2\sqrt{5}(2+3r^2) & 8r^2 \\ & 15+80r^2+444r^4 & 8r^2 & -2\sqrt{5}(3+16r^2) \\ & & 444+80r^2+15r^4 & -2\sqrt{5}r^2(16+3r^2) \\ & Symmetric & & 4(111+160r^2+111r^4) \end{pmatrix}. \qquad (17)$$

From

$$C^{trig}_{i,j} = \begin{pmatrix} 11.71 & -5.70 & -7.35 & 2.30 \\ & 131.95 & 2.30 & -25.38 \\ & & 184.41 & -23.74 \\ & Sym. & & 445.76 \end{pmatrix}$$

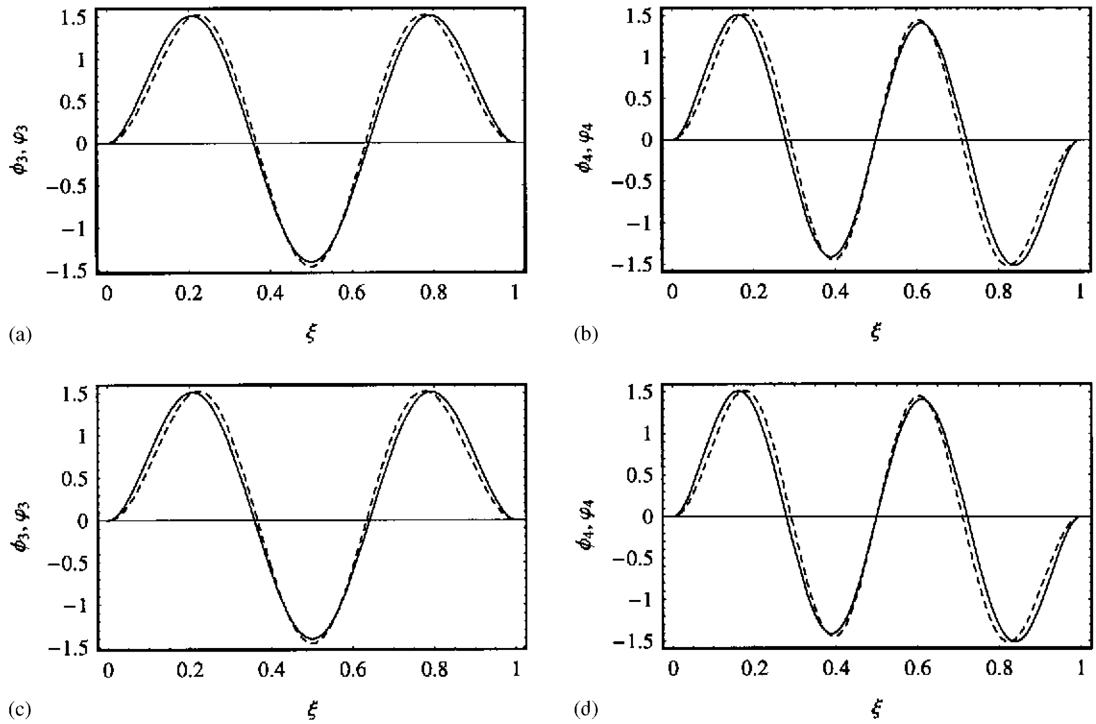

Fig. 1. Comparison of the clamped-beam modes: ——, true clamped-beam mode $\phi_i(\xi)$; – – – –, trigonometric clamped-beam mode $\varphi_i(\xi)$. (a) First modes; (b) second modes; (c) third modes; (d) fourth modes.

for $r = 0.9$, we find eigenvectors $\mathbf{e}_1 = (0.998, 0.0463, 0.0417, -0.00031)$, $\mathbf{e}_2 = (-0.0456, 0.996, -0.0137, 0.0794)$, $\mathbf{e}_3 = (-0.0417, 0.0046, 0.995, 0.0905)$, and $\mathbf{e}_4 = (0.00776, -0.080, -0.0896, 0.993)$ for the eigenvalues $\lambda = (11.14, 130.16, 182.56, 449.97)$. Hence, we have the approximation of $(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \approx (0.998\varphi_1(x)\varphi_1(y), 0.996\varphi_1(x)\varphi_3(y), 0.995\varphi_3(x)\varphi_1(y), 0.993\varphi_3(x)\varphi_3(y))$. By comparison, we see that $\{\phi_n(x)\phi_m(y)\}$ approximate the actual plate modes a little better than $\{\varphi_n(x)\varphi_m(y)\}$. However, since neither of them are the exact plate modes of a clamped plate, the analytical ease of $\varphi_n(\xi)$ far outweighs a slight improvement in the definition of plate modes formed by the true clamped-beam modes $\phi_n(\xi)$.

To sum up, there are two strategies for the Galerkin representation: either obtain modal equations for $\mathbf{q}$ from the pseudo-plate mode expansion (10) and then transform them into the modal equations for $\mathbf{Q}$ by Eq. (14), or derive directly the modal equations for $\mathbf{Q}$ from the actual plate mode expansion (15). In this paper, we resort to the former to parallel the previous analysis of simply supported plate [3].

4. Two plate equations for w and F

The w–F formulation is based on the transverse displacement equation (7) together with F, supplemented by compatibility relation (8). As pointed out in Section 3, we shall derive the modal equations for $\mathbf{q}$ by expanding transverse displacement w in $\mathbf{\Phi} = \{\phi_n(x)\phi_m(y)\}$ or $\{\varphi_n(x)\varphi_m(y)\}$, as indicated by Eq. (10). However, by conventional wisdom we represent the Airy stress function by

$$F = -\frac{C_y x^2}{2} - \frac{C_x y^2}{2r^2} + (1-v^2) \sum_{\substack{p=0,2,4,\ldots,M \\ (p=q\neq 0)}} \sum_{q=0,2,4,\ldots,M} f_{p,q} \cos(p\pi x)\cos(q\pi y), \tag{18}$$

in which the cosine sum extends to upper limit M. We call the first two right-hand side terms a homogeneous part F_h and the cosine sum the particular solution F_p. It is fair to say that representation (18) is Achilles' heel of the w–F formulation, because zero in-plane edge displacement conditions; i.e.,

$$u(x,y) = v(x,y) = 0, \tag{19}$$

at $x=0$ and 1 for all y and at $y=0$ and 1 for all x, cannot be translated into the equivalent boundary conditions for F. Hence, a way out of this predicament has been suggested by imposing certain boundary constraints on F in an average (integral) sense [4,5,13]. Of these, the following are relevant for the present discussion.

First of all, that no shear stresses exist around the plate edges is expressed on average by the integral constraints $\int_0^1 (\partial^2 F/\partial x\,\partial y)_{x=0,1}\,\mathrm{d}y = \int_0^1 (\partial^2 F/\partial x\,\partial y)_{y=0,1}\,\mathrm{d}x = 0$, and thereby justifying the cosine expansion of F_p. Secondly, zero in-plane displacements around the plate edges are expressed, again on average, by the integral constraints

$$\int_0^1\int_0^1 \left(\frac{1}{(1-v^2)}\left(r^2\frac{\partial^2 F}{\partial y^2} - v\frac{\partial^2 F}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\mathrm{d}x\,\mathrm{d}y = 0,$$
$$\int_0^1\int_0^1 \left(\frac{1}{(1-v^2)}\left(\frac{\partial^2 F}{\partial x^2} - vr^2\frac{\partial^2 F}{\partial y^2}\right) - \frac{r^2}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)\mathrm{d}x\,\mathrm{d}y = 0,$$

from which the constants in F_h are determined:

$$\begin{aligned} C_x = & -\tfrac{1}{2}\{(1+r^2 v)d_{1,1,0}q_1^2 + (d_{1,1,0} + r^2 v d_{3,3,0})q_2^2 \\ & + (r^2 v d_{1,1,0} + d_{3,3,0})q_3^2 + (1+r^2 v)d_{3,3,0}q_4^2 \\ & + 2r^2 v d_{1,3,0}q_1q_2 + 2d_{1,3,0}q_1q_3 + 2d_{1,3,0}q_2q_3 + 2r^2 v d_{1,3,0}q_3q_4\}, \end{aligned}$$

$$\begin{aligned} C_y = & -\tfrac{1}{2}\{(r^2+v)d_{1,1,0}q_1^2 + (v d_{1,1,0} + r^2 d_{3,3,0})q_2^2 \\ & + (r^2 d_{1,1,0} + v d_{3,3,0})q_3^2 + (r^2+v)d_{3,3,0}q_4^2 \\ & + 2r^2 d_{1,3,0}q_1q_2 + 2v d_{1,3,0}q_1q_3 + 2v d_{1,3,0}q_2q_3 + 2r^2 d_{1,3,0}q_3q_4\}, \end{aligned} \tag{20}$$

where $d_{i,j,k} = \int_0^1 \gamma_i'(\xi)\gamma_j'(\xi)\cos(k\pi\xi)\,\mathrm{d}\xi$. Note here and in what follows that $\gamma_i = \phi_i(\xi)$ or $\varphi_i(\xi)$ is depending on the choice of beam modes. Much has been argued for or against the use of Eq. (18) in the literatures [4,5,13], and we have nothing new to add to the controversy. We shall however take here a utilitarian view and judge the consequences of Eq. (18) a posteriori by the internal consistency of modal equations that are derived from it. Since F_h makes no contribution to the left-hand side of Eq. (8), $f_{p,q}$ are quadratic in q_n and hence, for instance, we have

$$f_{2,2} = \frac{r^2}{4\pi^4(1+r^2)^2}\{q_1^2(d_{1,1,2}^2 - c_{1,1,2}^2) + (q_2^2 + q_3^2)(d_{1,1,2}d_{3,3,2} - c_{1,1,2}c_{3,3,2}) \\ + q_4^2(d_{3,3,2}^2 - c_{3,3,2}^2) + q_1(q_2+q_3)(2d_{1,1,2}d_{1,3,2} - c_{1,1,2}c_{1,3,2} - c_{1,1,2}c_{3,1,2}) \\ + 2q_1q_4(d_{1,3,2}^2 - c_{1,3,2}c_{3,1,2}) + q_4(q_2+q_3)(2d_{1,3,2}d_{3,3,2} - c_{1,3,2}c_{3,3,2} - c_{3,1,2}c_{3,3,2}) \\ + q_2q_3(2d_{1,3,2}^2 - c_{1,3,2}^2 - c_{3,1,2}^2)\},$$

where $c_{i,j,k} = \int_0^1 \gamma_i(\xi)\gamma_j''(\xi)\cos(k\pi\xi)\,\mathrm{d}\xi$.

With the evaluation of C_x, C_y and $f_{p,q}$, we can carry out the Galerkin procedure by inserting Eqs. (10) and (18) into Eq. (7) and subsequently sorting out modal equations for $\mathbf{q}$. The upshot is then linear modal Eq. (11) to which is appended a myriad of cubic amplitude terms arising from the right-hand side of Eq. (7)

$$\ddot{\mathbf{q}} + \mathbf{Cq} + \mathbf{K} = 0, \tag{21}$$

where the components of $\mathbf{K} = (k_1, k_2, k_3, k_4)$ are cubic expressions:

$$k_1 = 4\kappa_1 q_1^3 + 3\kappa_5 q_1^2 q_2 + 3\kappa_6 q_1^2 q_3 + 2\kappa_{11} q_1 q_2 q_3 + \kappa_{12} q_2^2 q_3 \\ + \kappa_{13} q_2 q_3^2 + 2\kappa_{14} q_1 q_2^2 + 2\kappa_{15} q_1 q_3^2 \\ + 2\kappa_9 q_1 q_2 q_4 + 2\kappa_{10} q_1 q_3 q_4 + \kappa_{18} q_2 q_3 q_4 + 2\kappa_{19} q_1 q_4^2 \\ + \kappa_{21} q_2^3 + \kappa_{22} q_3^3 + \kappa_{25} q_2^2 q_4 + \kappa_{26} q_3^2 q_4 \\ + \kappa_{29} q_2 q_4^2 + \kappa_{30} q_3 q_4^2 + 3\kappa_{33} q_1^2 q_4 + \kappa_{34} q_4^3,$$

$$k_2 = 4\kappa_2 q_2^3 + \kappa_5 q_1^3 + 3\kappa_7 q_2^2 q_4 + \kappa_9 q_1^2 q_4 + \kappa_{11} q_1^2 q_3 + 2\kappa_{12} q_1 q_2 q_3 \\ + \kappa_{13} q_1 q_3^2 + 2\kappa_{14} q_1^2 q_2 + 2\kappa_{16} q_2 q_4^2 + \kappa_{18} q_1 q_3 q_4 + 2\kappa_{20} q_2 q_3^2 \\ + 3\kappa_{21} q_1 q_2^2 + 3\kappa_{23} q_2^2 q_3 + \kappa_{24} q_3^3 + 2\kappa_{25} q_1 q_2 q_4 + 2\kappa_{27} q_2 q_3 q_4 + \kappa_{28} q_3^2 q_4 \\ + \kappa_{29} q_1 q_4^2 + \kappa_{31} q_4^3 + \kappa_{35} q_3 q_4^2,$$

$$k_3 = 4\kappa_3 q_3^3 + \kappa_6 q_1^3 + 3\kappa_8 q_3^2 q_4 + \kappa_{10} q_1^2 q_4 + \kappa_{11} q_1^2 q_2 + \kappa_{12} q_1 q_2^2 \\ + 2\kappa_{13} q_1 q_2 q_3 + 2\kappa_{15} q_1^2 q_3 + 2\kappa_{17} q_3 q_4^2 \\ + \kappa_{18} q_1 q_2 q_4 + 2\kappa_{20} q_2^2 q_3 + 3\kappa_{22} q_1 q_3^2 + \kappa_{23} q_2^3 + 3\kappa_{24} q_2 q_3^2 \\ + 2\kappa_{26} q_1 q_3 q_4 + 2\kappa_{28} q_2 q_3 q_4 + \kappa_{30} q_1 q_4^2 \\ + \kappa_{32} q_4^3 + \kappa_{35} q_2 q_4^2 + \kappa_{27} q_2^2 q_4,$$

$$k_4 = 4\kappa_4 q_4^3 + \kappa_7 q_2^3 + \kappa_8 q_3^3 + \kappa_9 q_1^2 q_2 + \kappa_{10} q_1^2 q_3 + 2\kappa_{16} q_2^2 q_4 \\ + 2\kappa_{17} q_3^2 q_4 + \kappa_{18} q_1 q_2 q_3 + 2\kappa_{19} q_1^2 q_4 \\ + \kappa_{25} q_1 q_2^2 + \kappa_{26} q_1 q_3^2 + \kappa_{27} q_2^2 q_3 + \kappa_{28} q_2 q_3^2 + 2\kappa_{29} q_1 q_2 q_4 \\ + 2\kappa_{30} q_1 q_3 q_4 + 3\kappa_{31} q_2 q_4^2 + 3\kappa_{32} q_3 q_4^2 \\ + \kappa_{33} q_1^3 + 3\kappa_{34} q_1 q_4^2 + 2\kappa_{35} q_2 q_3 q_4. \tag{22}$$

Note that each k_i has exactly 20 (= 6!/3!3!) cubic terms, corresponding to the combinations with repetitions of forming $q_\ell q_m q_n$ out of $\mathbf{q}$. Besides, the coefficients $\kappa_1, \ldots, \kappa_{35}$ involve not only parameters r and v, but also the integrals $c_{i,j,k}$, $d_{i,j,k}$, $b_{i,j,k} = \int_0^1 \gamma_i(\xi)\gamma_j(\xi)\cos(k\pi\xi)\,\mathrm{d}\xi$, and $e_{i,j,k} = \int_0^1 \gamma_i(\xi)\gamma_j'(\xi)\sin(k\pi\xi)\,\mathrm{d}\xi$ which depend on the beam modes $\gamma_i = \phi_i(\xi)$ or $\varphi_i(\xi)$.

Let us denote by κ_n^{trig} the coefficients of k_i evaluated with trigonometric clamped-beam modes $\varphi_n(\xi)$, and present here only the following two as typical expressions:

$$\kappa_1^{trig} = \frac{8}{3}\left(1 + r^4 + 2r^2 v + \frac{4}{9} r^4 (1 - v^2)\left(\frac{17}{8}(1 + r^{-4}) + \frac{4}{(1 + r^2)^2} + \frac{1}{(4 + r^2)^2} + \frac{1}{(1 + 4r^2)^2}\right)\right)$$

and

$$\kappa_{10}^{trig} = \frac{32}{45\sqrt{5}}\left(30r^2(r^2 + v) + (1 - v^2)\left(224 + r^4\left(68 - \frac{100}{(1 + r^2)^2} + \frac{668}{(4 + r^2)^2} + \frac{671}{(1 + 4r^2)^2} + \frac{660}{(9 + r^2)^2} + \frac{39}{(9 + 4r^2)^2}\right)\right)\right).$$

Instead of the lengthy expressions, we present in Table 1 the numerical values of κ_n^{trig} for $r = 0.9$ and $v = \sqrt{0.1}$. Recall that $f_{p,q} = 0$ for p or $q > 6$ for the simply supported plate [3]. Here it turns out that $f_{p,q} = 0$ for p or $q > 8$, so that F_p is again a truncated cosine series for a clamped plate when we use trigonometric clamped-beam modes $\varphi_n(\xi)$. On the other hand, things are a little complicated when the true clamped-beam modes $\phi_n(\xi)$ are used. Since κ_n^{true} are too long to write out in detail, Table 2 summarizes their numerical values for $r = 0.9$, $v = \sqrt{0.1}$, and $M = 8$. More importantly, we find that $f_{p,q} \neq 0$ for all p and q under the true clamped-beam modes $\phi_n(\xi)$. Yet, the numerical values of $f_{p,q}$ are so small for p and $q > 8$ that there is hardly a noticeable difference in the numerical values of κ_n^{true} for $M = 8$ and 10, as shown in Tables 2 and 3, respectively.

For the internal consistency of cubic vector $\mathbf{K}$, we examine the Hamiltonian structure of Eq. (22). With the conjugate co-ordinates $\mathbf{p} = \dot{\mathbf{q}}$, the inertial term of Eq. (21) is related to kinetic energy $H_k = \frac{1}{2}\mathbf{p}^{\mathrm{T}}\mathbf{p}$ and the stiffness term to plate bending strain energy $H_b = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{C}\mathbf{q}$. We infer the membrane stretching strain energy by integration $\sum_{i=1}^{4} \int k_i \,\mathrm{d}q_i$ and, subsequently, elimination of

Table 1
Numerical values of κ_n^{trig} for $r = 0.9$, $v = \sqrt{0.1}$, and $M = 8$

$\kappa_1 = 10.46$,	$\kappa_2 = 230.16$,	$\kappa_3 = 337.18$,	$\kappa_4 = 573.22$,	$\kappa_5 = -28.57$,
$\kappa_6 = -32.23$,	$\kappa_7 = -354.50$,	$\kappa_8 = -496.47$,	$\kappa_9 = -82.06$,	$\kappa_{10} = -93.62$,
$\kappa_{11} = 92.67$,	$\kappa_{12} = -242.95$,	$\kappa_{13} = -250.85$,	$\kappa_{14} = 115.66$,	$\kappa_{15} = 141.68$,
$\kappa_{16} = 975.91$,	$\kappa_{17} = 1306.90$,	$\kappa_{18} = 484.31$,	$\kappa_{19} = 125.52$,	$\kappa_{20} = 477.85$,
$\kappa_{21} = -110.14$,	$\kappa_{22} = -160.02$,	$\kappa_{23} = 70.87$,	$\kappa_{24} = 93.36$,	$\kappa_{25} = 115.02$,
$\kappa_{26} = 171.04$,	$\kappa_{27} = -255.91$,	$\kappa_{28} = -372.45$,	$\kappa_{29} = -176.43$,	$\kappa_{30} = -263.39$,
$\kappa_{31} = -154.47$,	$\kappa_{32} = -93.70$,	$\kappa_{33} = 7.61$,	$\kappa_{34} = -5.30$,	$\kappa_{35} = -15.29$.

Table 2
Numerical values of κ_n^{true} for $r = 0.9$, $v = \sqrt{0.1}$, and $M = 8$

$\kappa_1 = 8.53$,	$\kappa_2 = 191.15$,	$\kappa_3 = 281.25$,	$\kappa_4 = 483.85$,	$\kappa_5 = -20.80$,
$\kappa_6 = -23.37$,	$\kappa_7 = -276.77$,	$\kappa_8 = -392.59$,	$\kappa_9 = -58.05$,	$\kappa_{10} = -67.54$,
$\kappa_{11} = 64.47$,	$\kappa_{12} = -175.48$,	$\kappa_{13} = -182.52$,	$\kappa_{14} = 90.62$,	$\kappa_{15} = 111.74$,
$\kappa_{16} = 785.82$,	$\kappa_{17} = 1055.87$,	$\kappa_{18} = 346.96$,	$\kappa_{19} = 102.25$,	$\kappa_{20} = 369.77$,
$\kappa_{21} = -77.75$,	$\kappa_{22} = -114.62$,	$\kappa_{23} = 42.59$,	$\kappa_{24} = 58.02$,	$\kappa_{25} = 75.97$,
$\kappa_{26} = 115.66$,	$\kappa_{27} = -153.63$,	$\kappa_{28} = -234.19$,	$\kappa_{29} = -121.92$,	$\kappa_{30} = -185.74$,
$\kappa_{31} = -96.45$,	$\kappa_{32} = -45.50$,	$\kappa_{33} = 5.18$,	$\kappa_{34} = -6.48$,	$\kappa_{35} = -22.56$.

Table 3
Numerical values of κ_n^{true} for $r = 0.9$, $v = \sqrt{0.1}$, and $M = 10$

$\kappa_4 = 484.41$,	$\kappa_7 = -276.80$,	$\kappa_8 = -392.61$,	$\kappa_{13} = -182.53$,	$\kappa_{16} = 786.04$,
$\kappa_{17} = 1056.03$,	$\kappa_{18} = 346.98$,	$\kappa_{20} = 369.78$,	$\kappa_{24} = 58.01$,	$\kappa_{25} = 75.96$,
$\kappa_{26} = 115.65$,	$\kappa_{27} = -153.58$,	$\kappa_{28} = -234.16$,	$\kappa_{29} = -121.88$,	$\kappa_{30} = -185.72$,
$\kappa_{31} = -96.53$,	$\kappa_{32} = -45.47$,	$\kappa_{34} = -6.51$,	$\kappa_{35} = -22.67$;	

The remaining κ_n are same as in Table 2 up to the first two decimal digits.

the redundant quartic terms [3]

$$\begin{aligned} H_m = {} & \kappa_1 q_1^4 + \kappa_2 q_2^4 + \kappa_3 q_3^4 + \kappa_4 q_4^4 + \kappa_5 q_1^3 q_2 + \kappa_6 q_1^3 q_3 + \kappa_7 q_2^3 q_4 \\ & + \kappa_8 q_3^3 q_4 + \kappa_9 q_1^2 q_2 q_4 + \kappa_{10} q_1^2 q_3 q_4 \\ & + \kappa_{11} q_1^2 q_2 q_3 + \kappa_{12} q_1 q_2^2 q_3 + \kappa_{13} q_1 q_2 q_3^2 + \kappa_{14} q_1^2 q_2^2 + \kappa_{15} q_1^2 q_3^2 \\ & + \kappa_{16} q_2^2 q_4^2 + \kappa_{17} q_3^2 q_4^2 + \kappa_{18} q_1 q_2 q_3 q_4 \\ & + \kappa_{19} q_1^2 q_4^2 + \kappa_{20} q_2^2 q_3^2 + \kappa_{21} q_1 q_2^3 + \kappa_{22} q_1 q_3^3 + \kappa_{23} q_2^3 q_3 \\ & + \kappa_{24} q_2 q_3^3 + \kappa_{25} q_1 q_2^2 q_4 + \kappa_{26} q_1 q_3^2 q_4 + \kappa_{27} q_2^2 q_3 q_4 \\ & + \kappa_{28} q_2 q_3^2 q_4 + \kappa_{29} q_1 q_2 q_4^2 + \kappa_{30} q_1 q_3 q_4^2 + \kappa_{31} q_2 q_4^3 \\ & + \kappa_{32} q_3 q_4^3 + \kappa_{33} q_1^3 q_4 + \kappa_{34} q_1 q_4^3 + \kappa_{35} q_2 q_3 q_4^2. \end{aligned}$$

From the total Hamiltonian $H(\mathbf{p}, \mathbf{q}) = H_k + H_b + H_m$, one immediately rederives Eq. (21) by Hamilton's equations of motion [14],

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{23}$$

hence the modal equations are Hamiltonian. Note that being Hamiltonian is a stronger dynamical property than merely energy conserving. In retrospect, the existence of the Hamiltonian is anticipated from the energy discussion in Section 2, whereby $H(\mathbf{p}, \mathbf{q})$ is nothing but the total plate energy $U_k + U_b + U_m$. First of all, we see from Eq. (4) that $U_k = H_k$ by inspection, because of the orthonormality of beam modes. We then show that $U_b = H_b$ by inserting Eq. (10) into Eq. (5) and similarly that $U_m = H_m$ upon introducing Eq. (18) into Eq. (9), after the indicated integrations are carried out. The algebraic tasks have been facilitated by symbolic manipulation software, such as

MATHEMATICA^{TM} [15]. In this paper, the existence of $H(\mathbf{p},\mathbf{q})$ will be taken as the a posteriori justification for the Airy function expansion (18), which is the cornerstone of the w–F formulation.

So far, we have discussed the Hamiltonian property of Eq. (21) which is the modal equation of pseudo-plate modes $\mathbf{\Phi}(x,y)$. However, for the modal equation of actual plate modes $\mathbf{\Theta}(x,y)$, we can transform $H(\mathbf{p},\mathbf{q})$ into $H(\mathbf{P},\mathbf{Q})=\frac{1}{2}\mathbf{P}^{\mathrm{T}}\mathbf{P}+\frac{1}{2}\mathbf{Q}^{\mathrm{T}}\boldsymbol{\lambda}\mathbf{Q}+H_m(\mathbf{Q})$ by Eq. (14), where $\mathbf{P}=\dot{\mathbf{Q}}$ is the new conjugate co-ordinates. Then, the modal equations for $\mathbf{Q}$ are derived from Hamilton's equations of motion

$$\dot{Q}_i=\frac{\partial H}{\partial P_i},\quad \dot{P}_i=-\frac{\partial H}{\partial Q_i}. \tag{24}$$

They are in fact identical to what one might have obtained by repeating the Galerkin procedure from alternate transverse displacement expansion (15).

5. Three plate equations for w, u, and v

Consistent with Eq. (10), we introduce the following expansions for in-plane displacements:

$$u=\sum_{i=2,4,6}\ \sum_{j=1,3,\dots,N} b_{i,j}\sigma_i(x)\sigma_j(y),\quad v=\sum_{i=1,3,\dots,N}\ \sum_{j=2,4,6} c_{i,j}\sigma_i(x)\sigma_j(y), \tag{25}$$

in which $\sigma_i(\xi)=\sqrt{2}\sin(i\pi\xi)$ are used to satisfy the zero plate edge conditions (19). With the four symmetric plate modes, the index i for u and j for v take on three values (2, 4, 6). On the other hand, the index j for u and i for v will take on all odd integers, hence we denote the upper summation limit by N. After introducing Eqs. (10) and (25) into Eqs. (3), we sort out algebraic equations for $b_{i,j}$ and $c_{i,j}$ by carrying out integrations with the factor $\sigma_n(x)\sigma_m(y)$. For a given N, there are $\frac{3}{2}(N+1)$ components each for $b_{i,j}$ and $c_{i,j}$, so that the simultaneous solution of Eqs. (3) requires inverting a matrix of $3(N+1)\times 3(N+1)$, which for instance is 6×6 for $N=1$. It is therefore not feasible to write down explicit expressions for $b_{i,j}$ and $c_{i,j}$, as we have presented an analytical expression for $f_{2,2}$ in Section 4. In any event, $b_{i,j}$ and $c_{i,j}$ are quadratic in q_n, so are (N_x, N_y, N_{xy}) by virtue of Eq. (2). The derivation of modal equations from Eq. (1) parallels what we have already done in Section 4 from Eq. (7). Consequently, the modal equations for $\mathbf{q}$ are the same as Eq. (21), but the components of $\mathbf{K}=(\mathfrak{R}_1,\mathfrak{R}_2,\mathfrak{R}_3,\mathfrak{R}_4)$ are now given by

$$\begin{aligned}\mathfrak{R}_1 =\ &\alpha_1^1q_1^3+\alpha_2^1q_1^2q_2+\alpha_3^1q_1q_2^2+\alpha_4^1q_2^3+\alpha_5^1q_1^2q_3+\alpha_6^1q_1q_2q_3\\ &+\alpha_7^1q_2^2q_3+\alpha_8^1q_1q_3^2+\alpha_9^1q_2q_3^2\\ &+\alpha_{10}^1q_3^3+\alpha_{11}^1q_1^2q_4+\alpha_{12}^1q_1q_2q_4+\alpha_{13}^1q_2^2q_4\\ &+\alpha_{14}^1q_1q_3q_4+\alpha_{15}^1q_2q_3q_4+\alpha_{16}^1q_3^2q_4+\alpha_{17}^1q_1q_4^2\\ &+\alpha_{18}^1q_2q_4^2+\alpha_{19}^1q_3q_4^2+\alpha_{20}^1q_4^3,\end{aligned}$$

$$
\begin{aligned}
\Re_2 = {} & \alpha_1^2 q_1^3 + \alpha_2^2 q_1^2 q_2 + \alpha_3^2 q_1 q_2^2 + \alpha_4^2 q_2^3 + \alpha_5^2 q_1^2 q_3 + \alpha_6^2 q_1 q_2 q_3 \\
& + \alpha_7^2 q_2^2 q_3 + \alpha_8^2 q_1 q_3^2 + \alpha_9^2 q_2 q_3^2 \\
& + \alpha_{10}^2 q_3^3 + \alpha_{11}^2 q_1^2 q_4 + \alpha_{12}^2 q_1 q_2 q_4 + \alpha_{13}^2 q_2^2 q_4 + \alpha_{14}^2 q_1 q_3 q_4 \\
& + \alpha_{15}^2 q_2 q_3 q_4 + \alpha_{16}^2 q_3^2 q_4 + \alpha_{17}^2 q_1 q_4^2 \\
& + \alpha_{18}^2 q_2 q_4^2 + \alpha_{19}^2 q_3 q_4^2 + \alpha_{20}^2 q_4^3,
\end{aligned}
$$

$$
\begin{aligned}
\Re_3 = {} & \alpha_1^3 q_1^3 + \alpha_2^3 q_1^2 q_2 + \alpha_3^3 q_1 q_2^2 + \alpha_4^3 q_2^3 + \alpha_5^3 q_1^2 q_3 \\
& + \alpha_6^3 q_1 q_2 q_3 + \alpha_7^3 q_2^2 q_3 + \alpha_8^3 q_1 q_3^2 + \alpha_9^3 q_2 q_3^2 \\
& + \alpha_{10}^3 q_3^3 + \alpha_{11}^3 q_1^2 q_4 + \alpha_{12}^3 q_1 q_2 q_4 \\
& + \alpha_{13}^3 q_2^2 q_4 + \alpha_{14}^3 q_1 q_3 q_4 + \alpha_{15}^3 q_2 q_3 q_4 + \alpha_{16}^3 q_3^2 q_4 + \alpha_{17}^3 q_1 q_4^2 \\
& + \alpha_{18}^3 q_2 q_4^2 + \alpha_{19}^3 q_3 q_4^2 + \alpha_{20}^3 q_4^3,
\end{aligned}
$$

$$
\begin{aligned}
\Re_4 = {} & \alpha_1^4 q_1^3 + \alpha_2^4 q_1^2 q_2 + \alpha_3^4 q_1 q_2^2 + \alpha_4^4 q_2^3 + \alpha_5^4 q_1^2 q_3 + \alpha_6^4 q_1 q_2 q_3 \\
& + \alpha_7^4 q_2^2 q_3 + \alpha_8^4 q_1 q_3^2 + \alpha_9^4 q_2 q_3^2 \\
& + \alpha_{10}^4 q_3^3 + \alpha_{11}^4 q_1^2 q_4 + \alpha_{12}^4 q_1 q_2 q_4 + \alpha_{13}^4 q_2^2 q_4 + \alpha_{14}^4 q_1 q_3 q_4 \\
& + \alpha_{15}^4 q_2 q_3 q_4 + \alpha_{16}^4 q_3^2 q_4 + \alpha_{17}^4 q_1 q_4^2 \\
& + \alpha_{18}^4 q_2 q_4^2 + \alpha_{19}^4 q_3 q_4^2 + \alpha_{20}^4 q_4^3.
\end{aligned} \tag{26}
$$

Note that each $\Re_i$ has exactly 20 cubic terms $q_\ell q_m q_n$ identical to those in k_i of Eq. (22), but coefficients α_m^n of the present *w–u–v* formulation are different from the *w–F* formulation. In this paper, we restrict ourselves to $N = 7$ in Eqs. (25). This therefore calls for inverting a 24×24 matrix for the solution of Eqs. (3) for $b_{i,j}$ and $c_{i,j}$, hence going beyond $N = 7$ would require excessive algebraic manipulations. We now denote by $\mathbf{K} = \{\Re_n^{true}\}$ the cubic vector for the true clamped-beam modes $\phi_i(\xi)$ and $\mathbf{K} = \{\Re_n^{trig}\}$ for trigonometric clamped-beam modes $\varphi_i(\xi)$. Only the numerical values of α_m^n are summarized in Tables 4 and 5 for $\mathbf{K} = \{\Re_n^{true}\}$ and $\mathbf{K} = \{\Re_n^{trig}\}$, respectively, evaluated under $r = 0.9$, $v = \sqrt{0.1}$, and $N = 7$. Let us now attempt to infer H_m directly from Eq. (26). Of the quartic terms generated by integration $\sum_{i=1}^4 \int \Re_i \, \mathrm{d}q_i$, the terms $q_\ell q_m q_n q_p$ do not at all share the same coefficient; for instance, we find $\alpha_4^1 q_2^3 q_1 \neq \frac{1}{3} \alpha_3^2 q_2^3 q_1$ from the numerical values of α_m^n given in Tables 4 and 5. For this reason, we cannot obtain H_m from $\sum_{i=1}^4 \int \Re_i \, \mathrm{d}q_i$ by eliminating the redundant quartic terms, as before in the *w–F* formulation. This therefore indicates that the modal equations with cubic vector $\mathbf{K} = \{\Re_n\}$ are not Hamiltonian, and hence do not conserve energy for the present *w–u–v* formulation with in-plane displacement expansions (25) truncated at $N = 7$.

To quantify departure from the Hamiltonian or energy-conservation property, let us now directly compute the membrane stretching strain energy by inserting Eq. (25) into Eq. (6) and carrying out the indicated integrations. We present first membrane stretching strain

Table 4
Numerical values for α_m^n of $\mathbf{K} = \Re_n^{true}$ for $r = 0.9$, $v = \sqrt{0.1}$, and $N = 7$

α_m^1 (m = 1–20)	34.77/ − 64.36/179.5/ − 96.27/ − 72.1/130.37/ − 175.14/220.53/ − 182.71/ − 143.69/ 14.53/ − 114.26/93.09/ − 134.34/342.12/144.08/201.43/ − 150.54/ − 227.16/ − 9.67
α_m^2 (m = 1–20)	−21.37/181.51/ − 276.93/594.24/65.12/ − 351.63/133.98/ − 182.42/740.39/67.18/ −58.11/178.97/ − 677.94/342.3/ − 340.2/ − 278.73/ − 145.92/1312.44/ − 33.42/ − 133.33
α_m^3 (m = 1–20)	−23.92/65/ − 174.62/47.37/223.47/ − 368.5/740.74/ − 412.95/192.44/868.31/ 68.6/344.37/ − 182.38/276.91/ − 526.12/ − 952.5/ − 219.45/ − 33.51/1722.1/ − 78.78
α_m^4 (m = 1–20)	4.77/ − 57.77/90.62/ − 218.6/ − 68.28/346.81/ − 171.11/140.33/ − 264.51/ − 308.89/ 203.34/ − 295.26/1270.97/ − 442.15/ − 63.76/1678.79/ − 30.11/ − 381.83/ − 224.99/1507.35

Table 5
Numerical values for α_m^n of $\mathbf{K} = \Re_n^{trig}$ for $r = 0.9$, $v = \sqrt{0.1}$, and $N = 7$

α_m^1 (m = 1–20)	43.73/ − 89.84/230.52/ − 107.53/ − 101.21/193.64/ − 218.19/299.97/ − 295.42/ − 169.83/ 23.4/ − 167.61/102.41/ − 198/501.73/192.17/256.66/ − 154.07/ − 300.62/ − 6.9
α_m^2 (m = 1–20)	−30.75/242.24/ − 336.36/573.94/99.72/ − 526.8/208.57/ − 272.97/1067.36/95.74/ −84.26/321.91/ − 685.44/512.45/ − 520.14/ − 382.07/ − 175.53/1354.32/ − 28.56/ − 176.46
α_m^3 (m = 1–20)	−32.76/91.94/ − 238.17/70.2/291.34/ − 500.95/924.38/ − 496.78/286.92/853.43/ 99.35/490/ − 255.43/349.91/ − 745.11/ − 1014.3/ − 274.44/ − 23.74/1818.98/ − 106.27
α_m^4 (m = 1–20)	7.81/ − 81.64/111.35/ − 226.63/ − 101.74/503.52/ − 250.53/177.74/ − 375.51/ − 328.38/ 256.89/ − 357.58/1314.7/ − 534.76/ − 49.54/1771.67/ − 23.64/ − 493.1/ − 337.15/1448.2

energy U_m^{true} for the true clamped-beam modes $\phi_i(\xi)$:

$$
\begin{aligned}
U_m^{true} = {} & 8.7q_1^4 + 244.91q_2^4 + 365.51q_3^4 + 591.44q_4^4 \\
& - 21.22q_1^3q_2 - 23.67q_1^3q_3 - 350.59q_2^3q_4 \\
& - 512.06q_3^3q_4 - 54.56q_1^2q_2q_4 - 71.37q_1^2q_3q_4 \\
& + 64.65q_1^2q_2q_3 - 177.32q_1q_2^2q_3 - 186.87q_1q_2q_3^2 \\
& + 94.24q_1^2q_2^2 + 117.17q_1^2q_3^2 + 956.04q_2^2q_4^2 \\
& + 1322.55q_3^2q_4^2 + 364.99q_1q_2q_3q_4 + 106.88q_1^2q_4^2 \\
& + 375.06q_2^2q_3^2 - 61.39q_1q_2^3 + 35.55q_2^3q_3 - 90q_1q_3^3 \\
& + 48.94q_2q_3^3 + 4.66q_1^3q_4 + 58.9q_1q_2^2q_4 \\
& - 124.61q_2^2q_3q_4 + 90.96q_1q_3^2q_4 - 191.62q_2q_3^2q_4 \\
& - 95.59q_1q_2q_4^2 - 142.95q_1q_3q_4^2 \\
& - 21.67q_2q_3q_4^2 - 5.72q_1q_4^3 - 67.28q_2q_4^3 - 21.7q_3q_4^3,
\end{aligned}
$$

and U_m^{trig} for trigonometric clamped-beam modes $\varphi_i(\xi)$:

$$\begin{aligned}
U_m^{trig} = {} & 10.71q_1^4 + 318.17q_2^4 + 472.46q_3^4 + 756.14q_4^4 \\
& - 29.31q_1^3q_2 - 32.9q_1^3q_3 - 478.14q_2^3q_4 \\
& - 689.08q_3^3q_4 - 83.9q_1^2q_2q_4 - 63.03q_1^2q_3q_4 \\
& + 93.61q_1^2q_2q_3 - 246.77q_1q_2^2q_3 - 254.55q_1q_2q_3^2 \\
& + 118.99q_1^2q_2^2 + 144.25q_1^2q_3^2 + 1261.75q_2^2q_4^2 \\
& + 1741.05q_3^2q_4^2 + 495.2q_1q_2q_3q_4 + 128.84q_1^2q_4^2 \\
& + 484.99q_2^2q_3^2 - 113.96q_1q_2^3 + 70.4q_2^3q_3 - 162.71q_1q_3^3 \\
& + 93.04q_2q_3^3 + 7.33q_1^3q_4 + 117.83q_1q_2^2q_4 \\
& - 258.13q_2^2q_3q_4 + 172.83q_1q_3^2q_4 - 370.69q_2q_3^2q_4 \\
& - 182.94q_1q_2q_4^2 - 262.37q_1q_3q_4^2 \\
& - 26q_2q_3q_4^2 - 7.28q_1q_4^3 - 149.26q_2q_4^3 - 90.53q_3q_4^3.
\end{aligned}$$

Note that the numerical coefficients of U_m^{true} and U_m^{trig} have been evaluated for $r = 0.9$, $v = \sqrt{0.1}$, and $N = 7$. According to Eq. (23), $\{\partial U_m/\partial q_i\}$ are components of the energy-conserving cubic vector $\mathbf{K}$. For instance, the first component $\partial U_m^{true}/\partial q_1 = 34.82q_1^3 - 63.67q_1^2q_2 + 188.48q_1q_2^2 \cdots -5.72q_4^3$ should be compared with $\Re_1^{true} = 34.77q_1^3 - 64.36q_1^2q_2 + 179.5q_1q_2^2 \cdots - 9.67q_4^3$ formed by α_m^n of Table 4. From the numerical coefficients of $\{\Re_i^{true}\}$ and $\{\partial U_m^{true}/\partial q_i\}$, one can estimate the deviations in cubic amplitude terms in the range (−41%, 263%). Similarly, the deviations in numerical coefficients of $\{\Re_i^{trig}\}$ and $\{\partial U_m^{trig}/\partial q_i\}$ are found in the range (−54%, 24%) when trigonometric clamped-beam modes $\varphi_i(\xi)$ are used. In Ref. [3], the deviations in the range (−140%, 94%) have been reported for a simply supported plate under $N = 7$, and we have shown that deviations become progressively smaller as the upper summation limit N is increased up to 35. It is however not feasible to present such evidence in the present clamped-plate case, due to the rapidly increasing algebraic manipulations beyond $N = 7$.

6. Conclusions

Recently, Geveci [6] attempted to compare the two formulations of *w–u–v* and *w–F* by numerically generated trajectories of the transverse displacement of a clamped thin plate. However, the assessment of complex trajectory patterns is highly subjective and hence inconclusive in the absence of a theoretical frame of reference for comparison. We have previously proposed [3] a metric for such comparison, whereby modal equations derived from the Galerkin procedure are required to conserve energy as dictated by first principles of mechanics. This paper extends the simply supported plate analysis [3] to a clamped plate. Although the details are a little more tedious here because the plate modes are no longer expressed in sine functions, we arrive at the same conclusion as before. That is, the modal equations of *w–F* formulation are Hamiltonian and hence energy conserving at any level of modal truncation, whereas modal equations of the *w–u–v* formulation cannot conserve energy when only 12 sine terms are retained in the in-plane displacement expansions. For the simply supported plate [3], we have shown that the energy conservation of *w–u–v* formulation improves successively as more and more sine terms are included in the in-plane displacement expansions. However, such successive analyses are not feasible for a clamped plate, because of the unwieldy algebraic manipulations beyond the 12 sine expansion terms of in-plane displacements.

References

[1] H.-N. Chu, G. Herrmann, Influence of large amplitudes on free flexural vibrations of rectangular elastic plates, *Journal of Applied Mechanics* 23 (1956) 532–540.
[2] C.-Y. Chia, *Nonlinear Analysis of Plates*, McGraw-Hill, New York, 1980 (Chapter 1).
[3] J. Lee, Comparison of the two formulations of *w–u–v* and *w–F* in nonlinear plate analysis, *Journal of Applied Mechanics* 69 (2002) 547–552.
[4] V.V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability*, MacMillan, New York, 1963.
[5] L. Librescu, *The Elastostatic and Kinetics of Anisotropic and Heterogeneous Shell-type Structures*, Noordhoff, Leiden, 1975 (Chapter I-7).
[6] B. Geveci, Flow Induced Nonlinear Vibrations of Rectangular Plates, PhD Dissertation in Mechanical Engineering and Mechanics, Lehigh University, 1999.
[7] R.H. Scanlan, R. Rosenbaum, *Introduction to the Study of Aircraft Vibration and Flutter*, Dover Publications, New York, 1951 (Chapter VII).
[8] N. Yamaki, Influence of large amplitudes on flexural vibrations of elastic plates, *Zeitschrift für Angewandte Mathematik und Mechanik* 41 (1961) 501–510.
[9] S. Maekawa, On the sonic fatigue life estimation of skin structures at room and elevated temperatures, *Journal of Sound and Vibration* 80 (1982) 41–59.
[10] C. Mei, D.B. Paul, Nonlinear multimode response of clamped rectangular plates to acoustic loading, *American Institute of Aeronautics and Astronautics Journal* 24 (1986) 643–648.
[11] K.D. Murphy, L.N. Virgin, S.A. Rizzi, The effect of thermal prestresses on free vibration characteristics of clamped rectangular plates: theory and experiment, *Journal of Vibration and Acoustics* 119 (1997) 243–249.
[12] J. Lee, Large-amplitude plate vibration in an elevated thermal environment, *Applied Mechanics Reviews* 46 (2) (1993) S242–254.
[13] E.H. Dowell, Nonlinear oscillations of a fluttering plate, *American Institute of Aeronautics and Astronautics Journal* 4 (1966) 1267–1275.
[14] H. Goldstein, *Classical Mechanics*, 2nd Edition, Addison-Wesley, Reading, MA, 1980 (Chapter 8).
[15] S. Wolfram, *The Mathematica Book*, 4th Edition, Wolfram Media, Wolfram Research, IL, 1999.